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LETTER TO THE EDITOR

Real space renormalization group analysis of the random field Ising model

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Abstract. We present an analytic Migdal–Kadanoff renormalization group analysis of the random field Ising model. The renormalization flows to a zero-temperature critical point, from which we calculate independently three critical exponents in arbitrary dimension. In three dimensions the magnetization exponent $\beta \approx 0.02$, and the Schwartz–Soffer inequality is almost satisfied as an equality. Expanding analytically in $\epsilon = d - 2$ we find that β and the distance from the upper bound of the equality go to zero exponentially with $1/\epsilon^2$.

After much confusion, a coherent picture of the phase transition in the three-dimensional random field Ising model (RFIM) is at last emerging (for reviews see for example [1–3]).

The Hamiltonian for the RFIM is

$$H = -J \sum_{\langle i,j \rangle} S_i S_j - \sum_i h_i S_i \qquad S_i = \pm 1 \tag{1}$$

where J is a ferromagnetic coupling constant ($J > 0$) and h_i is a random field at site i which we take to have a Gaussian distribution

$$P(h_i) = \frac{1}{\sqrt{2\pi} h} \exp\left[-\frac{(h_i - h_0)^2}{2h^2}\right] \tag{2}$$

with mean value h_0 and variance h.

The very existence of ferromagnetic order in the three-dimensional RFIM has been the subject of an intense debate. Using physical arguments, Imry and Ma [4] predicted $d_c = 2$. However, Parisi and Sourlas [5] found $d_c = 3$, following a more rigorous perturbation expansion about the upper critical dimension $d = 6$. It was therefore perhaps surprising to find convincing evidence of a transition from experiments on realizations of the RFIM such as the diluted antiferromagnet $Fe_xZn_{1-x}Cl_2$ in a magnetic field [6]. Subsequently, exact calculations showed that, for weak random fields, a magnetic phase is stable at low temperatures [7, 8]. There is, therefore, little doubt now that a phase transition does exist in three dimensions and interest has passed on to the details.

It has been suggested that the transition could be first order [9, 10]; however a growing body of work has recently emerged showing it to be second order, driven almost first order by the random fields [11–14].

The perturbation calculation [5] predicts 'dimensional reduction' with critical exponents in the presence of random fields equal to those for the pure model in two fewer dimensions: $d' = d - 2$. This implies only two independent exponents. However, in principle there

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are three: the transition is characterized by a zero-temperature fixed point in the variable $\omega = h/J$ [15]. As the system renormalizes to zero temperature, the disconnected part of the susceptibility is by itself divergent and characterized by exponent $\overline{\eta}$:

$$\chi^{\mathrm{dis}}(\boldsymbol{q}) = \left[\langle S_q\rangle\langle S_{-q}\rangle\right] \sim 1/q^{4-\overline{\eta}}. \tag{3}$$

Here $\langle\ldots\rangle$ represents the thermal average, and $[\ldots]$ the average over the random variables. Dimensional reduction gives $\overline{\eta} = \eta$, with η the usual susceptibility exponent. Phenomenological scaling arguments [16, 17], on the other hand, lead to a modified dimensional reduction $d' = d - \theta = d - 2 + \overline{\eta} - \eta$, where θ or $\overline{\eta}$ would correspond to a third independent exponent. An upper bound is given to this third exponent by the Schwartz–Soffer [18] inequality $\overline{\eta} \leqslant 2\eta$. The most reliable numerical estimates of the exponents come from series expansions [19], where one finds the inequality satisfied as an equality to the third significant figure. If this is rigorously true one returns, *de facto*, to a situation with two independent exponents.

In this letter we present an analytic real space renormalization group (RSRG) calculation using the Migdal–Kadanoff (MK) technique. We calculate the critical exponents in arbitrary dimension by expanding the renormalization equations close to the zero-temperature fixed point [15]. In three dimensions we find a value for the magnetization exponent $\beta \approx 0.02$. The Schwartz–Soffer [18] inequality is almost satisfied as an equality and from a development in $\epsilon = d - 2$ we find that both β and $\overline{\eta} - 2\eta$ tend to zero exponentially with $1/\epsilon^2$.

We begin with a decimation procedure for a one-dimensional chain, which we then generalize to dimensions-d using the Migdal–Kadanoff approximation. Evaluating the trace over alternate spins on an N spin chain, the relevant part of the new partition function for an $N/2$ spin chain can be expressed in the form

$$Z_{i-1,i+1} = \delta \exp\beta[J'_{i-1,i+1}S_{i-1}S_{i+1} + h'_{i-1}S_{i-1} + h'_{i+1}S_{i+1}] \tag{4}$$

where $\beta = 1/k_{\mathrm{B}}T$ and δ is a constant. The transformations for $J'_{i-1,i+1}$ and h'_i in terms of the initial variables are given by

$$\begin{aligned}
J'_{i-1,i+1} &= \frac{1}{4\beta}\log\left(\frac{\cosh\beta(J_{i-1,i}+J_{i,i+1}+h_i)\cosh\beta(J_{i-1,i}+J_{i,i+1}-h_i)}{\cosh\beta(-J_{i-1,i}+J_{i,i+1}+h_i)\cosh\beta(J_{i-1,i}-J_{i,i+1}+h_i)}\right)\\
h'_{i+1} &= h_{i+1} + H_{i+1,-1} + H_{i+1,+1}\\
H_{i+1,\sigma} &= \frac{1}{4\beta}\log\left(\frac{\cosh\beta(J_{i+1,i+1+\sigma}+J_{i+1+\sigma,i+1+2\sigma}+h_{i+1+\sigma})}{\cosh\beta(J_{i+1,i+1+\sigma}+J_{i+1+\sigma,i+1+2\sigma}-h_{i+1+\sigma})}\right)\\
&\quad+\frac{1}{4\beta}\log\left(\frac{\cosh\beta(J_{i+1,i+1+\sigma}-J_{i+1+\sigma,i+1+2\sigma}+h_{i+1+\sigma})}{\cosh\beta(-J_{i+1,i+1+\sigma}+J_{i+1+\sigma,i+1+2\sigma}+h_{i+1+\sigma})}\right).
\end{aligned} \tag{5}$$

The transformation does not maintain the initial conditions, and after a single iteration the initially constant exchange parameters develop random components. The field h_i on the decimated site is shared between sites $i \pm 1$ in the new space through $H_{i-1,+1}$ and $H_{i+1,-1}$, which introduces correlations between $J'_{i-1,i+1}$ and $h'_{i\pm1}$ [13]. We are not able to treat these correlations analytically in a satisfactory manner, and we are forced to approximate the transformation for further iterations by replacing $J_{i,j}$ on the right-hand side of equation (5) by the first moment of the distribution $J = \overline{J_{i,j}}$ evaluated at the previous iteration. Under this approximation the fluctuations of the random fields are underestimated. We rectify this by replacing $H_{i+1,-1} + H_{i+1,+1}$ in equation (5) by $2H_{i+1,-1}$, whereby h'_{i+1} depends on h_i and h_{i+1} but no longer depends on h_{i+2}.

To understand this further one should consider the renormalization of the random fields in the absence of correlations and as the bond strength J goes to zero. Rescaling by a factor b involves replacing an element of volume b^d by a single point in a renormalized space. Under these conditions the rescaled field variance is given by uncorrelated fluctuations of the random field within the volume element, $h' = b^{d/2}h$. If the random fields $h_i, h_{i+2}, \ldots$ from the decimated spins are shared between two different sites ($i-1$ and $i+1$, $i+1$ and $i+3, \ldots$) without correlations being taken into account, then the new field are of the form $h'_{i+1} = h_{i+1} + 1/2(h_i + h_{i+2})$, which gives a field variance $h' = \sqrt{3/2}h$; less than the value $\sqrt{2}h$ imposed by dimensionality. We see that without the neglected correlations the fluctuations of the random field would be smoothed over and the effect of the random field implicitly underestimated. If the fields are repartitioned as proposed above, one immediately finds an upper bound for the renormalized field, $h'_{i+1} = h_{i+1} + h_i$, which gives the correct field variance $h' = \sqrt{2}$. This is the value one would find from the exact decimation in the limit as the coupling constants vanish, if the correlations were treated correctly. This series of approximations is best tested *a postori*.

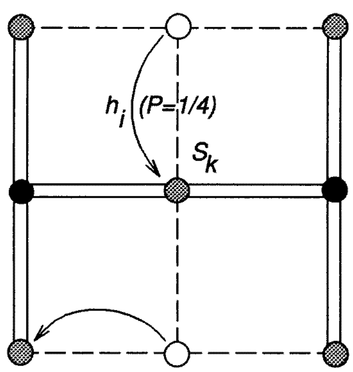

Figure 1. The Migdal–Kadanoff scheme with field partition, in two dimensions. Bonds are displaced from the dotted to the double lines. The fields h_i on the removed sites (open circles) are placed on one of the four sites, k, with probability $P = \frac{1}{4}$.

The Migdal–Kadanoff [20] scheme poses equally delicate, and related problems for the random field. A two-dimensional symmetric bond moving scheme, which lends itself well to analytic work, is shown in figure 1. The field on the central spin must be partitioned among the remaining sites, as in the renormalization 2^d field elements must contribute to the single point in the new space. In any symmetric partitioning, for example placing $\frac{1}{4}$ of the field h_i on each of the sites, k, one is faced with the same problem as before: the fluctuations of the random field are smoothed over, and its effect is underestimated. The correct procedure is, rather, to attribute the central field to any one of the sites k with probability $P = \frac{1}{4}$. To see this one must again consider dimensional arguments for the random fields in the limit of zero coupling constants. As any site, k, can receive fields from two neighbouring cells, the total field h_k^* is either the original field only, the sum of two fields, or of three fields, with probabilities $P = \frac{9}{16}$, $\frac{3}{8}$, or $\frac{1}{16}$, respectively. The variance of the distribution for the field on site k is therefore $(h^*)^2 = 3/2h^2$. Each renormalized cell contains two fields of the type h_k^*, plus a field from a corner site, so we have the total field variance $h'^2 = h^2 + 2(h^*)^2 = 4h^2$. Thus we find the correct upper bound for the rescaled random fields.

The procedure also works in three dimensions [21] where we find $(h^*)^2 = 7/3h^2$, after which we are able to propose the generalization to to the d dimensions:

$$(h^*)^2 = \frac{2^d - 1}{d} h^2. \tag{6}$$

We emphasize that, with this choice for the field partitioning our calculation predicts $d_c = 2$, in agreement with Imry–Ma arguments. For any field partition that underestimates the random field fluctuations we find $1 < d_c < 2$. This is a general result. In fact one can work backwards, imposing Imry–Ma arguments at the outset and arriving at equation (6) in the approximation where the $J_{i,j}$ are constant.

Following the bond moving scheme, we replace J by $\alpha_d J$, $\alpha_d = 2^{d-1}$, on the right-hand side of equations (5). We then average the terms involving J and h_i over a field distribution with variance h^*. Our d-dimensional decimation equations finally read

$$\begin{aligned}
J' &= \frac{1}{2\beta}\int_{-\infty}^{\infty} \mathrm{d}t\; P(t) \log \frac{\cosh \beta(2\alpha_d J + t)}{\cosh \beta t} \\
h'^2 &= h^2 + \frac{d}{4\beta^2}\int_{-\infty}^{\infty} \mathrm{d}t\; P(t) \log^2 \frac{\cosh \beta(2\alpha_d J + t)}{\cosh \beta(2\alpha_d J - t)} \\
h'_0 &= h_0 + \frac{d}{2\beta}\int_{-\infty}^{\infty} \mathrm{d}t\; P(t) \log \frac{\cosh \beta(2\alpha_d J + t)}{\cosh \beta(2\alpha_d J - t)} \\
P(t) &= \frac{1}{\sqrt{2\pi}\, h^*} \exp - \frac{(t - h_0^*)^2}{2h^{*2}} \qquad h_0^* = \frac{2^d - 1}{d} h_0.
\end{aligned} \tag{7}$$

The natural variables for discussing the renormalization are T/J and $\omega = h/J$, together with the constant magnetic field h_0. The zero-field fixed point is unstable to disorder and the ferromagnetic and paramagnetic phases are separated by a critical line, with all trajectories near the phase boundary flowing towards a new 'zero-temperature' fixed point at $T/J = 0$ [15] with $\omega = \omega_c$. The value ω_c and the critical exponents can be found by expanding the equations (7) at $T = 0$. After some tedious manipulation we find

$$\begin{aligned}
&J' = \alpha_d J \phi(\omega^*/\sqrt{2}\alpha_d) \\
&\omega'^2 J'^2/J^2 = \omega^2\{1 + (2^d - 1)\,\mathrm{erf}(\alpha_d\sqrt{2}/\omega^*)\} + 4d\alpha_d^2\{1 - \mathrm{erf}(\alpha_d\sqrt{2}/\omega^*)\} \\
&\qquad\qquad -2\sqrt{\frac{2}{\pi}}\, d\alpha_d\omega^* \exp\left(-\frac{2\alpha_d^2}{\omega^{*2}}\right) \\
&h'_0 = h_0\{1 + (2^d - 1)\,\mathrm{erf}(\alpha_d\sqrt{2}/\omega^*)\} \\
&\phi(x) = 2\int_0^{1/x} \frac{\mathrm{d}t}{\sqrt{\pi}} \exp(-t^2)(1 - xt)
\end{aligned} \tag{8}$$

where $\mathrm{erf}(x)$ is the error function and $\omega^* = h^*/J$. The function $\phi(\omega)$ is positive for $\omega \geqslant 0$, it decreases as ω increases, and is contained within the interval [0,1].

Eliminating J and J' in equations (8) and setting $\omega = \omega' = \omega_c$ we find an implicit equation for the critical field which can be solved numerically. In three dimensions $\omega_c = 1.956$, and we recuperate the correct lower critical dimension, $d_c = 2$ by setting $\omega_c = 0$. Close to the fixed point the renormalization equations for a change of scale b take the form

$$(h_0/T)' = b^x(h_0/T) \qquad (T/J)' = b^{-y}(T/J) \qquad t' = b^z t \qquad t = \omega - \omega_c \tag{9}$$

with x, y, and z all positive [15]. Putting equations (8) and (9) equal and linearizing with respect to t, T/J, and h_0 we solve numerically for x, y, z in d dimensions [21]. Using the scaling relations for the RFIM [15]

$$\nu = 1/z \qquad 2-\alpha = (d-y)\nu \qquad \beta = \nu(d-x)$$
$$\gamma = (2x-y-d)\nu \qquad \delta = (x-y)/(d-x)$$
$$\eta = d+2+y-2x \qquad \bar{\eta} = d+4-2x \tag{10}$$

we find the following complete set of exponents in three dimensions:

$$\omega_c = 1.956 \qquad x = 2.991 \qquad y = 1.491 \qquad z = 0.449$$
$$\nu = 2.23 \qquad \beta = 0.02 \qquad \alpha = -1.360 \qquad \gamma = 3.318$$
$$\eta = 0.510 \qquad \bar{\eta} = 1.019 \qquad \bar{\eta} - 2\eta = -0.002. \tag{11}$$

Our results are in very close agreement with stochastic Migdal–Kadanoff renormalization results [13, 14], which is rather encouraging. Most notably, in stochastic approaches one is able to retain the fluctuations induced in the coupling constants by the random fields. From our calculation it seems that these fluctuations are almost exactly compensated for, at least within the Migdal–Kadanoff approximation, by our field exchange procedure.

Our value of $\beta = 0.02$ can be compared with $\beta = 0.05$ from the energy minimization scheme of Ogielski [11], $\beta = 0.02$ from stochastic Migdal–Kadanoff [13] and $\beta = 0.00 \pm 0.05$ from Monte Carlo simulation [22]. With four different approaches giving similar small values for the magnetization exponent we begin to get an established picture of the the random field disorder driving the transition to the limit of being first order.

We gain more insight into the general trends for the exponents by expanding analytically in powers of $\epsilon = d-2$, where to first order in ϵ we recover the analytic results of Cao and Machta [13].

$$x = d = 2+\epsilon \qquad y = d/2 = 1+\epsilon/2 \qquad z = d/2-1 = \epsilon/2. \tag{12}$$

These differ slightly from the first calculation by Bray and Moore [15], as one finds $\nu = 2/\epsilon$, compared with their value of $\nu = 1/\epsilon$. To order ϵ we, therefore, find $\bar{\eta} - 2\eta = 0$ and two independent parameters only.

One can show that the leading contribution to both β and $\bar{\eta} - 2\eta$ is exponential in $1/\epsilon^2$. After some calculation we find

$$\beta = \frac{3}{4}\exp\left(-\frac{4}{\pi\epsilon^2\log^2 2}\right) \qquad \bar{\eta} - 2\eta = -\frac{3\pi\epsilon^3\log^2 2}{32}\exp\left(-\frac{4}{\pi\epsilon^2\log^2 2}\right) \tag{13}$$

which explains the consistently small values in the literature.

All numerical work gives the Schwartz–Soffer inequality satisfied as an equality [13, 14, 19, 22] within the numerical errors and from series expansions one finds $2\eta = \bar{\eta}$ in $d = 3$, $d = 4$, and $d = 5$. However, in a replica calculation Mezard and Young predict a solution for $\bar{\eta} - 2\eta < 0$ with values lying in a narrow range close to zero, but which they are unable to calculate. In our scheme, which does not suffer from statistical error, we are able to quantify the margin by which the equality is not satisfied. We remark that with this exponential dependence, our results are consistent with both the high-precision series expansion and the replica calculation.

In figure 2 we show η and $\bar{\eta}$, together with $\bar{\eta} - 2\eta$ against d. In contrast to the series expansion, the difference, $\bar{\eta} - 2\eta$, begins to grow between $d = 3$ and $d = 4$. This could indicate a development towards the 'dimensional reduction' result $\bar{\eta} = \eta$ at $d = 6$ [12]. However, η and $\bar{\eta}$ do not fall smoothly towards zero as one approaches $d = 6$, rather they reach minimum values between $d = 4$ and $d = 5$. This indicates the breakdown of the Migdal–Kadanoff scheme at higher dimension, as one might expect from results on the pure system.

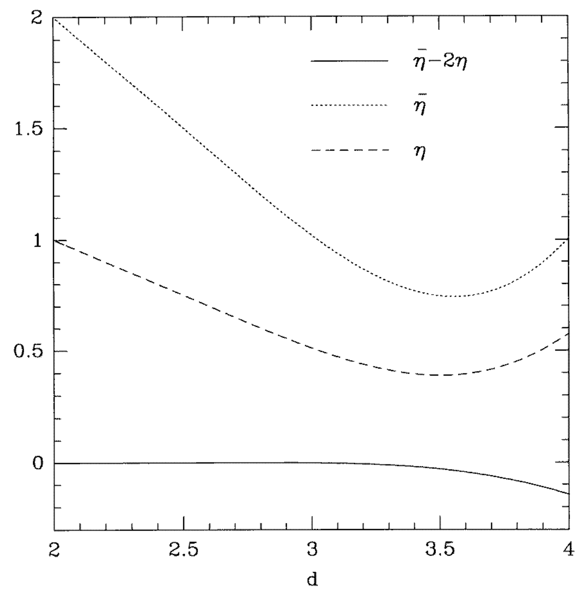

Figure 2. Exponents η, $\bar{\eta}$, and the difference $\bar{\eta} - 2\eta$ against the dimension d.

In this letter we have shown that the exponents of the random field Ising model can be calculated analytically with precision, following a series of three physically motivated approximations. The most notable of these is the neglect of the disorder induced into the renormalized coupling constants by the random fields. Within the framework of this approximation, two subsequent steps impose the correct upper bound for the random field fluctuations in the renormalized space. We find excellent agreement with stochastic methods, and in addition are able to directly address the question of the number of independent exponents. We find that the deviation from two-parameter scaling is exponentially small in $\epsilon = d - 2$, and find the same exponential dependence for the magnetization exponent β.

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References

[1] Nattermann T and Villain J 1988 *Phase Trans.* **11** 817
[2] Belanger D P and Young A P 1991 *J. Magn. Magn. Matter* **100** 272
[3] Belanger D P 1992 *Recent Progress in Random Magnets* ed D H Ryan (Singapore: World Scientific) p 277 (experiment)
Shapir Y 1992 *Recent Progress in Random Magnets* ed D H Ryan (Singapore: World Scientific) p 309 (theory)
[4] Imry Y and Ma S-K 1975 *Phys. Rev. Lett.* **35** 1399
[5] Parisi G and Sourlas N 1979 *Phys. Rev. Lett.* **43** 744
[6] Belanger D P, King A R, Jaccarino V and Cardy J L 1983 *Phys. Rev.* B **28** 2522
[7] Imbrie J Z 1984 *Phys. Rev. Lett.* **53** 1747
[8] Bricmont J and Kupiainen A 1987 *Phys. Rev. Lett.* **59** 1829
[9] Young A P and Nauenberg M 1958 *Phys. Rev. Lett.* **54** 2429
[10] Houghton A, Khurana A and Seco F J 1985 *Phys. Rev. Lett.* **55** 856
[11] Ogielski A T 1986 *Phys. Rev. Lett.* **57** 1251
[12] Mézard M and Young A P 1992 *Europhys. Lett.* **18** 653

[13] Cao M S and Machta J 1993 *Phys. Rev.* B **48** 3177
[14] Dayan I, Schwartz M and Young A P 1993 *J. Phys. A: Math. Gen.* **26** 3093
[15] Bray A J and Moore M A 1985 *J. Phys.: Condens. Matter* **18** L927
[16] Villain J 1985 *J. Physique* **46** 1843
[17] Fisher D S 1986 *Phys. Rev. Lett.* **56** 416
[18] Schwartz M and Soffer A 1985 *Phys. Rev. Lett.* **55** 2499
[19] Gofman M, Adler J, Aharony A, Harris A B and Schwartz M *Phys. Rev. Lett.* **71** 1569
[20] Migdal A A 1993 *Sov. Phys.–JETP* **42** 743 1976
[21] Fortin J-Y and Holdsworth P C W 1996 to be published
[22] Rieger H 1995 *Ann. Rev. Comput. Phys.* **2** 296